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# Structure selection for global vector field reconstruction by using the identification of fixed points

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Global vector field reconstruction is a well established technique to provide phenomenological models from nonlinear data, in particular when all information is contained in a so-called standard function. In the case when the standard function is taken as a ratio of polynomials, we establish that information about the fixed points of the system can be automatically retrieved from the data, allowing one to build a better model by selecting an appropriate structure. The method is exemplified in the case of the variable $z$ of the Rössler system, which constitutes a rather acid test case. [S1063-651X(99)06008-0]

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## I. INTRODUCTION

Although the original phase space of a nonlinear dynamical system is not necessarily accessible from measurements (in particular, when a single variable is recorded), pioneering papers [1–3] established that a reconstructed phase space may be built and, also, the idea that it is possible to retrieve a set of equations from deterministic data has been introduced. These landmark papers led to the development of so-called global vector field reconstructions in which, starting from a scalar recorded time series for continuous time systems, phenomenological models can be obtained under the form of a set of differential equations. Such reconstructions have actually been successfully obtained, and validated, both from numerical and, later on, from experimental systems [4–13].

In this paper, we consider the case where global vector field reconstructions rely on derivative coordinates, i.e., the phase space is spanned by the observable and its derivatives, up to a certain order. All the information relevant to the nonlinear dynamics is then contained in a so-called standard function. Originally, the standard function was taken as a rational function ([6], and references therein), but this choice exhibited a lack of robustness, so that it has been given up to the profit of polynomial expansions [6–13].

Our aim here is to demonstrate that a time series allows one to determine the fixed points underlying the data, which is the main result of this paper. This information provides the opportunity for model structure selection [14] and, as a by-product, allows one to use again rational functions in a more robust way, extending the range of models usable for global vector field reconstructions. These ideas will be applied to the case of the variable $z$ of the Rössler system which, as we shall see, provides a very acid test case.

The paper is organized as follows. Section II demonstrates, in the case of a rational function model (which actually generalizes the use of polynomial expansions), how information about the fixed points of the dynamical system can be retrieved from a time series. In Sec. III, we apply this new technique to the case of the variable $z$ of the Rössler system and explain why it constitutes an acid test case. Section IV is a conclusion.

## II. STRUCTURE INFORMATION FROM DATA

Let us consider a nonlinear dynamical system defined by a set of autonomous ordinary differential equations:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x};\boldsymbol{\mu}), \tag{1}$$

in which $\boldsymbol{x}(t) \in \mathbb{R}^n$ is a vector valued function depending on a parameter $t$ called the time and $\boldsymbol{f}$, the so-called vector field, is an $n$-component smooth function generating a flow $\phi_t$. $\boldsymbol{\mu} \in \mathbb{R}^m$ is the parameter vector with $m$ components.

The system (1), called the original system, may be unknown. For convenience, we present the method with $n=3$. Following a standard unfavorable hypothesis, only one variable is assumed to be known. Let this observable be called $x$. The aim is hereafter to reconstruct a vector field equivalent to the original one under the form of a so-called standard system built on the observable and on its successive derivatives according to

$$\begin{aligned} \dot{X} &= \dot{x} = Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= F_s(X,Y,Z), \end{aligned} \tag{2}$$

in which the reconstructed state space of the standard system is spanned by derivative coordinates $(X,Y,Z)=(x,\dot{x},\ddot{x})$ and $F_s$ is the so-called standard function to be evaluated.

A satisfactory global vector field reconstruction is achieved if a good enough approximation $\tilde{F}_s$ to $F_s$ is designed. In this paper, we assume a model structure for $\tilde{F}_s$ under the form of a ratio of polynomials reading as

$$\tilde{F}_s = \frac{Q(X,Y,Z)}{D(X,Y,Z)} = \frac{\sum_{p=1}^{N_Q} Q_p P^p}{\sum_{p=1}^{N_D} D_p P^p}, \tag{3}$$

where $N_Q$ and $N_D$ designate the number of monomials $P^p$ involved in the numerator polynomial $Q(X,Y,Z)$ and in the denominator polynomial $D(X,Y,Z)$, respectively. Also, the notation $P^p$ designates monomials reading as

$$P^p = X^i Y^j Z^k \tag{4}$$

with a biunivocal relationship between natural numbers $p$ and triplets $(i,j,k)$ defined in Ref. [7], and later generalized to the case of $n$-uplets [12].

Fixed points are among the most important invariants to be correctly reproduced by any modeling technique [14]. In the particular case of the standard system of Eq. (2), all coordinates of the fixed points are equal to zero except the first one, designated by $X_p$, which must be a zero of the standard function when the other coordinates are set to zero. Therefore, if $\{S_p\}_{p=1}^{N_S}$ designates the set of fixed points, we have

$$S_p \begin{cases} F_s(X_p,0,0)=0, \\ Y=0, \\ Z=0. \end{cases} \tag{5}$$

It is then required that the reconstructed standard system possess the same fixed points as the exact standard system. Consequently, the reconstructed standard function must have the same zeros as the exact standard function, when all the variables $Y$ and $Z$ are set to zero. From Eq. (3), we then must have

$$Q(X,Y,Z) = \prod_{p=1}^{N_S} (X-X_p) \prod_{u=1}^{N_u} (X-X_u) + \overline{Q}(X,Y,Z), \tag{6}$$

$$D(X,Y,Z) = D(X) \prod_{u=1}^{N_u} (X-X_u) + \overline{D}(X,Y,Z). \tag{7}$$

There, $\overline{Q}(X,Y,Z)$ is a contribution to the numerator $Q(X,Y,Z)$, which does not contain its constant term nor any monomial depending only on the variable $X$. By Eqs. (5), it must be zero when $Y$ and $Z$ are zero. Similarly, $\overline{D}(X,Y,Z)$ is a contribution to the denominator $D(X,Y,Z)$, which does not contain its constant term nor any monomial depending only on the variable $X$. $D(X)$ is a polynomial depending only on the variable $X$. $X_p$ are again the values of the variable $X$ associated to the fixed points. The quantities $X_u$ also cancel the numerator when $Y$ and $Z$ are zero, but they are not associated with any fixed point. Instead, they simplify with similar terms in the denominator when $Y$ and $Z$ are zero. $N_u$ designates the number of $X_u$ values.

To generate a well determined approximation problem for the structure exhibited in Eqs. (6) and (7), it is first necessary to evaluate the quantities $X_p$ and $X_u$ from the data. This demand relies on the evaluation of the numerator as follows.

In the reconstructed phase space $(X,Y,Z)$, the standard function may be estimated by using $\dot{Z}$ [Eq. (2)], which is the third-order derivative of the observable $x$. Let us then choose $(N_T-1)$ vectors of the reconstructed phase space for which the standard function $\dot{Z}=\dddot{x}$ is zero. These data generate the following system of equations:

$$\frac{\sum_{p=1}^{N_Q} Q_p P_1^p}{\sum_{p=1}^{N_D} D_p P_1^p} = 0,$$

$$\frac{\sum_{p=1}^{N_Q} Q_p P_i^p}{\sum_{p=1}^{N_D} D_p P_i^p} = 0, \tag{8}$$

$$\frac{\sum_{p=1}^{N_Q} Q_p P_{N_T-1}^p}{\sum_{p=1}^{N_D} D_p P_{N_T-1}^p} = 0,$$

in which $P_i^p (i=1,\ldots,N_T-1)$ are monomials evaluated by using the vectors $(X,Y,Z)_i, i=1,\ldots,N_T-1$. Since the denominators are generically not zero, the set of equations (8) makes sense and can be simplified to

$$\sum_{p=1}^{N_Q} Q_p P_1^p = 0,$$

$$\sum_{p=1}^{N_Q} Q_p P_i^p = 0, \tag{9}$$

$$\sum_{p=1}^{N_Q} Q_p P_{N_T-1}^p = 0.$$

The system (9) is a system of $(N_T-1)$ equations for $N_Q$ unknown coefficients $Q_p$. When the number of equations is equal to the number of unknowns, it can be solved by an inversion technique. We may also generate an overdetermined system and solve it in the least-square sense. In particular, beside the $(N_T-1)$ vectors involved in the systems (8) or (9), we use an $N_T$th vector for which $\dot{Z}=\dddot{x}$ is not zero, generating an extra equation:

$$\frac{\sum_{p=1}^{N_Q} Q_p P_{N_T}^p}{\sum_{p=1}^{N_D} D_p P_{N_T}^p} = \dot{Z}. \tag{10}$$

From Eq. (3), this equation may be rewritten as

$$\frac{\sum_{p=1}^{N_Q} Q_p P_{N_T}^p}{D(X,Y,Z)_{N_T}} = \dot{Z}, \tag{11}$$

which may be appended to the system (8). We then introduce renormalized coefficients $\tilde{Q}_p$, associated with $Q_p$ according to

$$\tilde{Q}_p = Q_p / D(X,Y,Z)_{N_T}. \tag{12}$$

We can solve the system (8) supplemented by Eq. (11) in the least-square sense to determine the unknown coefficients $\tilde{Q}_p$.

Afterward, we consider Eq. (6) which, for $Y=Z=0$, leads to

$$Q(X,0,0)=\prod_{p=1}^{N_S}(X-X_p)\prod_{u=1}^{N_u}(X-X_u). \tag{13}$$

From Eqs. (3), (11), and (12), and recalling that $\tilde{F}_s=\dot{Z}$, we obtain

$$Q(X,Y,Z)=D(X,Y,Z)_{N_T}\sum_{p=1}^{N_Q}\tilde{Q}_p P_{N_T}^p(X,Y,Z), \tag{14}$$

which may be specified to

$$Q(X,0,0)=D(X,0,0)_{N_T}\sum_{p=1}^{N_Q}\tilde{Q}_p P_{N_T}^p(X,0,0), \tag{15}$$

in which $P^p(X,0,0)$ is the monomial $P^p$ taken for $Y=Z=0$. By using the bijective relationship between natural numbers $p$ and triplets $(i,j,k)$, see Eq. (4), these monomials are actually monomials $X^{i(p)}$ depending only on the variable $X$ in which $i$ depends on $p$ such as $i\neq 0$ and $j=k=0$. Therefore, from Eqs. (13) and (15), the values of $X_p$ and $X_u$ are evaluated by solving

$$\sum_{p=1}^{N_Q}\tilde{Q}_p(X^{i(p)})=0. \tag{16}$$

The solutions of Eq. (16) can be obtained, for instance, by using Laguerre's method [15]. Let us also remark that solving Eq. (16) provides us with both $X_p$ values associated to fixed points and $X_u$ values, which are not associated to fixed points.

Next, we have from Eq. (3)

$$\sum_{p=1}^{N_Q}Q_pP^p-\dot{Z}\sum_{p=1}^{N_D}D_pP^p=0, \tag{17}$$

while from Eq. (6)

$$\sum_{p=1}^{N_Q}Q_pP^p=\prod_{p=1}^{N_S}(X-X_p)\prod_{u=1}^{N_u}(X-X_u)+\sum_{p=1}^{N_Q}Q_pP_c^p, \tag{18}$$

in which $P_c^p$ are monomials $P^p$ where the powers of $Y$ and $Z$ are not simultaneously equal to zero; i.e., by Eq. (4), this means $j\neq 0$ or/and $k\neq 0$.

Therefore, once $X_p$ and $X_u$ are determined as explained above, the approximation problem subsequently consists in the determination of the coefficients $D_p$ and $Q_p$ according to the following equation:

$$\sum_{n=1}^{N_Q}Q_nP_c^n-\dot{Z}\sum_{p=1}^{N_D}D_pP^p=-\prod_{p=1}^{N_S}(X-X_p)\prod_{u=1}^{N_u}(X-X_u), \tag{19}$$

in which indices $n$ are such as $j\neq 0$ or/and $k\neq 0$.

If a good enough reconstruction is achieved, the values $X_u$ can be identified among the double set of values $X_p$ and $X_u$ by numerically solving

$$\sum_{p=1}^{N_Q}D_p(X^{i(p)})=0, \tag{20}$$

in which $i(p)$ has the same meaning as in the similar Eq. (16). The values $X_p$ which are coordinates of the fixed points of the system are then simultaneously obtained.

Therefore, our method yields the locations of the fixed points of a global model with a specific structure (when the standard function is a ratio of polynomials), using the data to determine its coefficients. In utmost rigor, the fixed points so obtained are also the fixed points of the system underlying the data only if the structure of the approximated standard function is the same as the structure of the exact (in general unknown) standard function. Also, strictly speaking, Eq. (17) is fulfilled only approximately, since the model might be, in general, inaccurate and since $\dot{Z}$ is a third derivative. However, under the assumptions that the model is the correct one and ignoring the noise, our algorithm allows one to determine the fixed points of the system from the numerator polynomial directly from the data and constitutes the main result of this paper. We shall comment later on situations where the aforementioned assumptions are not exactly satisfied.

## III. APPLICATION TO THE RÖSSLER SYSTEM

### A. Generalities

In this section, we consider the well known original Rössler system reading as

$$\begin{aligned}\dot{x}&=-y-z,\\ \dot{y}&=x+ay,\\ \dot{z}&=b+z(x-c),\end{aligned} \tag{21}$$

with a phase space spanned by original coordinates $(x,y,z)$ and in which $(a,b,c)$ are the control parameters. It is then easily demonstrated that, if $y$ is the observable, the standard function takes a polynomial form and can then be easily approximated by a polynomial expansion [7], furthermore ensuring a diffeomorphism between the original phase space $(x,y,z)$ and the reconstructed phase space $(y,\dot{y},\ddot{y})$ [16]. Conversely, observables $x$ and $z$ provide transformations between original and reconstructed phase spaces which are only almost everywhere diffeomorphic, i.e., they form a diffeomorphism but for a set of Lebesgue measure zero. As can be readily checked, the existence of these sets is associated with the fact that the corresponding standard functions have a structure of rational functions ([17], and references therein). When the variable $x$ is used, the standard function possesses a first-order singularity and the corresponding rational function can be well approximated by a polynomial structure [7], a fact which may be understood by invoking the Weierstrass convergence theorem [18]. When the variable $z$ is used, the standard function now possesses a second-order singularity and the corresponding standard function

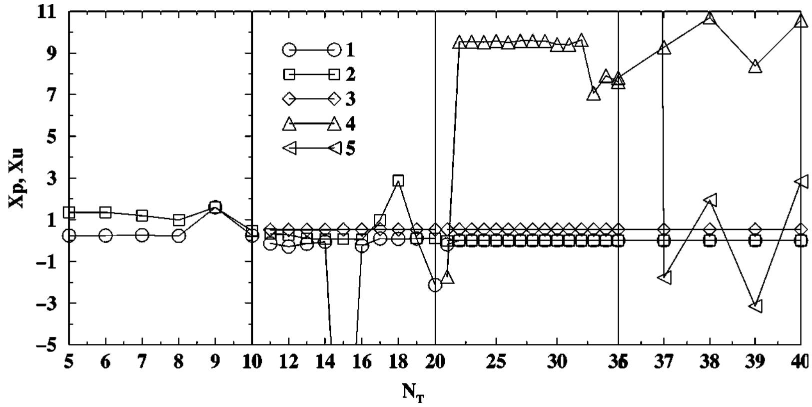


FIG. 1. $X_p$ and $X_u$ values plotted versus $N_T$.

can no longer be well approximated by a polynomial expansion, leading to a failure in global vector field reconstruction with such an expansion [7]. Therefore, there is an ordering $y > x > z$, meaning that $y$ is the best observable while $z$ is the worst observable. This also means that our ability to observe the state of a system may depend on the recorded variable. This fact has been exemplified in Refs. [16,19,20] and receives a quantitative confirmation in Ref. [17] which introduces an observability index. According to this observability index, we obtain an ordering $y > x > z$ which is the same as the one relying on the order of singularities in the standard function. In Ref. [7], it has been stated that the variable $z$ of the Rössler system constituted an acid test case for global vector field reconstructions. The above observations provide a sound basis to explain why it is so.

To the best of our knowledge, the only valuable global model obtained from the variable $z$ of the Rössler system has been built by using a 4D phase space [17], while 3D models have only been successful by using observables $x$ and $y$. The present work takes into account the fact that, if the standard function is a ratio of polynomials as initially used by Gouesbet ([6], and references therein), its singularities come from the denominator and thus can be distinguished from the fixed points of the system, which are zeros of the numerator. The determination of fixed points, as explained in Sec. II, then will allow one to successfully provide a global vector field reconstruction with rational functions, in a 3D-phase space, for the variable $z$ of the Rössler system.

For the exact standard system derived from the $z$ variable of the Rössler system, two fixed points $S_+$ and $S_-$ are obtained given by

$$S_\pm \begin{cases} X_\pm = \dfrac{c \pm (c^2 - 4ab)^{1/2}}{2}, \\ Y = 0, \\ Z = 0. \end{cases} \tag{22}$$

Again with the variable $z$ of the Rössler system taken as the observable, the exact standard function, denoted $F_z$, reads as

$$F_z = b - cX - Y + aZ + aX^2 - XY + \frac{(ab + 3Z)Y - aY^2 - bZ}{X} + \frac{2bY^2 - 2Y^3}{X^2} \tag{23}$$

exhibiting the aforementioned second-order singularity which, in practice, cannot be easily handled by a polynomial approximation. This standard function can alternatively be given the form of a rational function reading as

$$F_z = (2X^2 + abXY - 2bXZ + 2bY^2 - cX^3 - X^2Y + aX^2Z - aXY^2 + 3XYZ - 2Y^3 + aX^4 - X^3Y)/X^2. \tag{24}$$

This structure has previously been used in Ref. [6] and references therein, but without any success. It will be successful in this paper by extracting fixed-point information from the data and using it as a means for structure selection.

### B. Extracting fixed points and the model

The chaotic data series that we use is generated from the numerical integration of Eq. (21) with the control parameters $(a,b,c) = (0.398, 2.0, 4.0)$ and with a time step $h$ equal to 0.001. Since the pseudoperiod is equal to about 6.2, the data series has 6200 points per pseudoperiod. Successive derivatives up to the third order are estimated from the data. To this purpose, a sixth degree interpolating polynomial is built, centered at each point where the derivatives are evaluated, by using the six nearest neighbors. Derivatives are then obtained by deriving these polynomials. The window size $\tau$ is taken to be equal to 7 in terms of $h$.

The first step of the reconstruction process consists in the determination of the $X_u$ and $X_p$ values. This is achieved by using $N_T$ between 5 and 40 with $N_Q$ equal to $N_T$, i.e., we use $(N_T - 1)$ vectors for building Eqs. (8), plus one vector for Eq. (11). For each value of $N_T$, the coefficients $\tilde{Q}_p$ are computed by solving Eq. (8) supplemented with Eq. (11) in the least-square sense and $X_u$ and $X_p$ values are determinated by solving Eq. (16) with Laguerre's method [15].

For a given value of $N_Q$, the number of $X_u$ and $X_p$ values is equal to the highest power in the variable $X$ involved in Eq. (16), i.e., to the greatest value of $i$ in the triplets $(i,j,k)$ that are related to natural numbers $p$ according to the bi-univocal relationship defined in Ref. [7] with $1 \leqslant p \leqslant N_Q$. The minimal value for $N_T$ is chosen to be 5, because, beneath 5, the system would only have linear terms in the numerator and then could not be rich enough to model the considered nonlinear process. The obtained values are plotted versus $N_T$ in Fig. 1. If a value is complex with an imaginary part not negligible with respect to its real part, only its

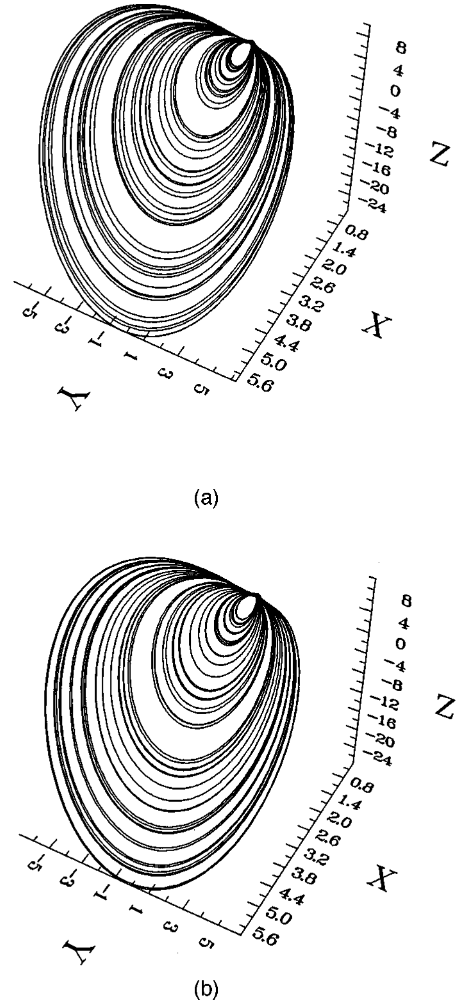


FIG. 2. (a) Trajectory obtained from the numerical integration of the exact standard system calculated from the $z$ variable of the Rössler system for the control parameters $(a,b,c) = (0.398,2.0,4.0)$. (b) Trajectory obtained from the numerical integration of the reconstructed model.

TABLE I. Coefficients of the standard function: $D_p^e$ and $Q_p^e$ are the values of the coefficients of the exact standard function $F_s$; $D_p$ and $Q_p$ are, respectively, the estimated values of $D_p^e$ and $Q_p^e$; in the last column are the triplets corresponding to the index $p$ of the coefficients.

| $p$ | $D_p^e$ | $D_p$ | $(i,j,k)$ |
|---|---|---|---|
| 1 | 0.0 | $0.32944495601098\times 10^{-4}$ | (0,0,0) |
| 2 | 0.0 | $-0.887\,385\,716\,978\,10\times 10^{-4}$ | (1,0,0) |
| 3 | 0.0 | $-0.982\,851\,441\,497\,10\times 10^{-4}$ | (0,1,0) |
| 4 | 0.0 | $0.523\,536\,279\,044\,98\times 10^{-5}$ | (0,0,1) |
| 5 | 1.0 | 1.000 000 000 000 00 | (2,0,0) |
| | $Q_p^e$ | $Q_p$ | |
| 1 | 0.0 | $0.368\,668\,847\,348\times 10^{-5}$ | (0,0,0) |
| 2 | 0.0 | $0.737\,309\,247\,675\,02\times 10^{-5}$ | (1,0,0) |
| 3 | 0.0 | $-0.170\,987\,031\,830\,36\times 10^{-4}$ | (0,1,0) |
| 4 | 0.0 | $-0.100\,216\,142\,529\,33\times 10^{-3}$ | (0,0,1) |
| 5 | 2.0 | 2.000 427 991 773 627 | (2,0,0) |
| 6 | $ab=0.796$ | 0.796 043 224 012 91 | (1,1,0) |
| 7 | $-2b=-2.0$ | −1.999 671 242 686 3 | (1,0,1) |
| 8 | $2b=4.0$ | 3.999 896 371 185 6 | (0,2,0) |
| 9 | 0.0 | $-0.597\,091\,322\,752\,88\times 10^{-4}$ | (0,1,1) |
| 10 | 0.0 | $-0.109\,413\,866\,921\,11\times 10^{-3}$ | (0,0,2) |
| 11 | $-c=-4.0$ | −4.000 700 161 440 0 | (3,0,0) |
| 12 | −1.0 | −0.999 876 401 143 07 | (2,1,0) |
| 13 | $a=0.398$ | 0.397 551 480 154 17 | (2,0,1) |
| 14 | $-a=-0.398$ | −0.397 415 044 978 10 | (1,2,0) |
| 15 | 3.0 | 2.999 770 329 837 9 | (1,1,1) |
| 16 | 0.0 | $-0.199\,002\,432\,436\,83\times 10^{-4}$ | (1,0,2) |
| 17 | −2.0 | −1.999 787 834 348 4 | (0,3,0) |
| 18 | 0.0 | $0.342\,676\,016\,903\,93\times 10^{-4}$ | (0,2,1) |
| 19 | 0.0 | $0.577\,930\,207\,944\,06\times 10^{-5}$ | (0,1,2) |
| 20 | 0.0 | $-0.657\,719\,101\,694\,41\times 10^{-6}$ | (0,0,3) |
| 21 | $a=0.398$ | 0.397 796 806 766 65 | (4,0,0) |
| 22 | −1.0 | −1.000 089 673 920 3 | (3,1,0) |

modulus is plotted. If the imaginary part is negligible, the real part is plotted.

For $N_T$ between 5 and 10, the value labeled 1 is essentially constant while the second one decreases until $N_T=9$, where both of them become complex. For $N_T$ between 11 and 20, three values are obtained. The two first values are sometimes oscillating but the third one remains constant and can be estimated to be equal to $0.52\pm 0.02$. For $N_T$ between 21 and 35, four values are obtained. For $N_T$ between 22 and 35, the third one is remarkably stable and is equal to $0.527\,706\pm 0.000\,002$. In the same interval, the first two values remain real and are constant around $0.\pm 0.002$, with sometimes a very small imaginary part not larger than 0.002. For $N_T$ between 22 and 32, the fourth one remains very stable at $9.50\pm 0.15$. After $N_T=35$ and until 40, the number of values is 5. For $N_T$ between 35 and 40, the first three values are the same as before but the fourth one becomes slightly oscillating around 9.0. The fifth one is clearly unstable. It then seems reasonable to only retain the four $X_p$ and $X_u$ values that remain very stable for $N_T$ between 22 and 32 (in fact, we retain the average of these values on this interval).

Using these values, a good approximation to the standard function is easily found with the driving vector $(N_q,h,N_s,N_Q,N_D,\tau)=(100,0.001,4,22,5,7)$ in which (i) $N_q$ is the number of vectors $(X_i,Y_i,Z_i,\dot{Z}_i)(i\in[1,N_q])$ on the net, (ii) $h$ is the time step between each of them, (iii) $N_s$ is the number of quadruplets $(X_i,Y_i,Z_i,\dot{Z}_i)$ sampled per pseudoperiod, (iv) $N_Q$ and $N_D$ are the number of coefficients of the numerator and of the denominator, respectively, and (v) $\tau$ is the window size on which the derivatives are estimated. Indeed, the integration of the reconstructed model provides an attractor which may be shown to be topologically equivalent to the original Rössler attractor obtained from the numerical integration of Eqs. (2) with $F_s=F_z$ as given in Eq. (24) (see Fig. 2). From the denominator coefficients $D_i$ and solving Eq. (20), the $X_u$ values may be computed and identified among the $X_p$ and $X_u$ values as explained in Sec. II. They appear to identify with the two values labeled 1 and 2 in Fig. 1, now denoted $X_{u1}$ and $X_{u2}$, and are equal to $0.0\pm 0.002$. We may then identify afterward the coordinates of the two fixed points $X_{p1}$ and $X_{p2}$ of the model, which are equal to $0.527\,706\pm 0.000\,002$ and 9.50

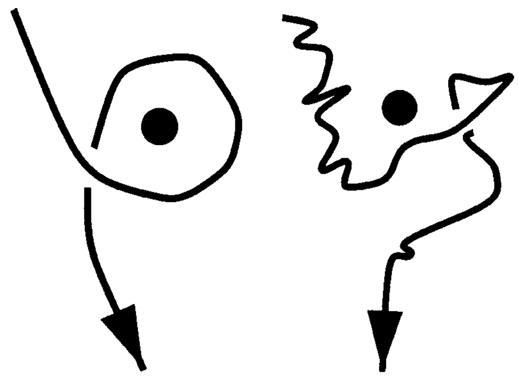


FIG. 3. A small perturbation may crucially change the structure of the flow in the neighborhood of a fixed point. When the data are noisy, the fixed points may become difficult to identify as exemplified in the sketch (b). In such a case, a small perturbation may move the fixed point location.

$\pm 0.15$, respectively. These values compare very favorably with the exact numerical values of $X_-$ and $X_+$, which are, respectively, equal to 0.527 708 4 and 9.522. The approximation to $F_s$ then reads as

$$\tilde{F}_s = [(X-X_{p1})(X-X_{p2})(X-X_{u1})(X-X_{u2}) + Q_3 Y + Q_4 Z + Q_6 XY + Q_7 XZ + Q_8 Y^2 + Q_9 YZ + Q_{10} Z^2 + Q_{12} X^2 Y + Q_{13} X^2 Z + Q_{14} XY^2 + Q_{15} XYZ + Q_{16} XZ^2 + Q_{17} Y^3 + Q_{18} Y^2 Z + Q_{19} YZ^2 + Q_{20} Z^3 + Q_{22} X^3 Y]/(D_1 + D_2 X + D_3 Y + D_4 Z + D_5 X^2), \tag{25}$$

where the values of the coefficients $Q_n$ and $D_n$ are reported in Table I.

Next, since, in the present case, we know the original standard system, we may compare our model to its exact analytically derived form.

From the $X_p$ and $X_u$ values and from the evaluation of the coefficients $Q_n$ with index $n$ such as $j \neq 0$ or $k \neq 0$, and of the coefficients $D_p$, it is possible to evaluate all $Q_p$'s, $p \in [1, N_Q]$, and all $D_p$'s, $p \in [1, N_D]$ for the approximation to the standard function. These coefficients are reported in Table I together with the exact values denoted $Q_p^e$ and $D_p^e$. The relative error $\epsilon$ between the exact coefficients and the estimated coefficients is defined according to

$$\epsilon = \frac{\sum_{i=1}^{N_Q} |Q_i - Q_i^e| + \sum_{i=1}^{N_D} |D_i - D_i^e|}{\sum_{i=1}^{N_Q} |Q_i^e| + \sum_{i=1}^{N_D} |D_i^e|} \tag{26}$$

and is found to be less than 0.02%, which is indeed a satisfactory result.

### C. Complementary discussion

We finally introduce some comments promised at the end of Sec. II. First of all, when the model takes the form of a polynomial expansion, it can nevertheless produce satisfactory models even when the exact standard function does not possess a polynomial form, including when the data are of an experimental nature [8,9,11,21]. This fact mathematically relies on the Weierstrass convergence theorem [18]. There is no similar theorem for rational functions but, by enlarging the class of models available for global vector field reconstructions, it is likely that experimental data can sometimes be successfully reconstructed by using rational functions rather than by using polynomials. Nevertheless, numerical instabilities are only avoided in the case where the right structure is selected, i.e., when fixed points are determined. One significant issue concerns the robustness of the present method against noise perturbations. Even if the computation of successive derivatives amplifies the noise, it has already been observed that a satisfactory model may be obtained from noisy experimental data when the fourth time derivative is involved in the reconstruction of a 4D model [21]. In the present work, however, problems associated with noise still arise for the fixed-point extraction.

In particular, as sketched in Fig. 3, a small perturbation in the neighborhood of one fixed point may be sufficient to locally change the flow structure preventing us from correctly identifying the fixed-point location. A remedy to this problem is still to be found to extend the range of applicability of the present new method for fixed-point identification and structure selection.

## IV. CONCLUSION

By using rational functions, it is shown that the class of dynamical systems for which a reconstructed model may be obtained is extended to a wider class. To this purpose, the structure of the standard function involved in the reconstructed model is selected by distinguishing the fixed-point coordinates from other zeros of the function. In this way, a successful 3D model is obtained starting from the $z$ variable of the Rössler system, which was remaining an acid test case only already solved with a 4D differential embedding. A structure selection for the reconstructed standard function has therefore allowed one to improve the quality of the model by reducing its complexity. Moreover, an appropriate structure for a rational standard function prevents us from numerical difficulties during the integration process. Nevertheless, the fixed-point extraction process is sensitive to noise perturbations, an issue to be solved to allow one to use our method for realistic experimental data.

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